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Coding Algorithm for Grayscale Images Based on Piecewise Uniform Quantizers

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Abstract. In this paper, a piecewise uniform quantizer for input samples with discrete amplitudes for Laplacian source is designed and analyzed, and its forward adaptation is done. This type of quantizers is very often used in practice for the purpose of compression and coding of already quantized signals. It is shown that the design and the adaptation of quantizers for discrete input samples are different from the design and the adaptation of quantizers for continual input samples. A weighting function for *PSQNR* (peak signal-to-quantization noise ratio), which is obtained based on probability density function of variance of standard test images is introduced. Experiments are done, applying these quantizers for compression of grayscale images. Experimental results are very well matched to the theoretical results, proving the theory. Adaptive piecewise uniform quantizer designed for discrete input samples gives for 9 to 20 dB higher *PSQNR* compared to the fixed piecewise uniform quantizer designed for discrete input samples. Also it is shown that the adaptive piecewise uniform quantizer designed for discrete input samples gives higher *PSQNR* for 1.46 to 3.45 dB compared the adaptive piecewise uniform quantizer designed for continual input samples, which proves that the discrete model is more appropriate for image quantization than continual model.
Keywords: piecewise uniform quantizer, input samples with discrete amplitude, forward adaptation, grayscale image.

1. Introduction

Quantization is one of the main steps in digitalization of analog signals (Jayant and Noll, 1984). Very often, quantization is done in two phases. In the first phase, quantization with large number of levels is done, with the aim of A/D conversion. In such way quantized samples, which have discrete amplitudes, are further quantized in the second phase, using quantizer with much smaller number of levels, with the aim of compression. Quantizers in the first and in the second phase are different, because the quantizer in the first phase has continual input samples (i.e., samples with continual amplitudes) while the quantizer in the second phase has discrete input samples (i.e., samples with discrete amplitudes). Quantizers in the first phase, for the continual input samples, have been analyzed in many papers (Na, 2001, 2008). The aim of this paper is the design and adaptation of the quantizer in the second phase, for discrete input samples. In the paper Savic *et al.* (2010), the uniform quantizer for discrete input samples was considered.

In this paper, the piecewise uniform quantizer for discrete input samples is considered, for signals with Laplacian distribution. The piecewise uniform quantizer (Nikolic *et al.*, 2011) joins good characteristics of both uniform and nonuniform quantizers: it can achieve *SQNR* (signal-to-quantization noise ratio) near to *SQNR* of the nonuniform quantizer but with much less complexity. The piecewise uniform quantizer can be considered as generalized quantized whose special cases are the uniform and the nonuniform quantizer. It will be shown that the design of the quantizer for discrete input samples is different from the design of the quantizer for continual input samples, because discrete samples have bounded amplitudes and therefore the overload distortion does not exist.

The forward adaptation (Nikolic and Peric, 2008) of the piecewise uniform quantizer for discrete input samples is also done in this paper. It is shown that the adaptation of the quantizer for discrete input samples is different from the adaptation of the quantizer for continual input samples. Namely, the adaptation of the quantizer for discrete input samples limited in amplitude should be done only if one condition (that the maximal amplitude of the adaptive quantizer is smaller than the maximal amplitude of the fixed quantizer) is fulfilled, while the adaptation of quantizers for continual input samples can be done without any conditions.

Experiments are done, applying the previously defined fixed and adaptive piecewise uniform quantizers for discrete input samples for grayscale image compression. Three standard test images are analyzed (Lena, Street and Boat). The grayscale image (whose pixels have discrete integer values) is divided into blocks, for each block the mean value is calculated and the difference between the pixel value and the quantized mean value of the block is quantized using the piecewise uniform quantizer for discrete input samples. This quantizer can be applied because the difference has integer (i.e., discrete) values with Laplacian distribution (Jayant, 1984; Salomon, 2007). The standard deviation of the difference, denoted as σ_d, is calculated. A weighted function for σ_d, obtained experimentally, based on three previously mentioned test images, is introduced in this paper. As a measure of quality of the reconstructed image, *PSQNR* (peak signal-to-quantization noise rat io) is used (Sayood, 2006). *PSQNR* is calculated for all values of σ_d. The average value of *PSQNR* is calculated in two ways: by classic averaging (where each value of σ_d has the same weight) and by weighted averaging (where the weighting function for σ_d is used).

It is shown by experiment that the adaptive piecewise uniform quantizer designed for discrete input samples gives for 9 to 20 dB higher *PSQNR* compared to the fixed piecewise quantizer designed for discrete input samples. Also, the adaptive quntizer designed for discrete input samples gives for 1.46 to 3.45 dB higher *PSQNR* compared to the adaptive quantizer designed for continual input samples. It proves that the discrete model is more appropriate for image quantization than continual model.

It is shown that experimentally obtained *PSQNR* (averaged using the weighted function for σ_d) is matched very well with *PSQNR* theoretically obtained. In this way, theoretical analysis for the fixed and the adaptive piecewise uniform quantizers for discrete input samples is proved. Also, the validity of the introducing of the weighted function for σ_d is proved. Using the adaptive piecewise quantizer, the near lossless compression is achieved.

This piecewise uniform quantizer for discrete input samples can be used, besides for the image compression, also for compression of all other signals with Laplacian distribution.

2. Construction of Fixed Piecewise Uniform Quantizer for Discrete Input

Quantization is often done in two phases. In the first phase, samples with continual amplitude are quantized with the uniform quantizer Q_0 with N_0 output levels $X = \{x_1, \dots, x_{N_0}\}$. Continual amplitudes of the input samples are described with some probability density function (pdf), denoted with $p(x)$. In this paper, we will use Laplacian pdf, which is defined as $p(x) = (\frac{1}{\sqrt{2}\sigma})\exp(-\frac{\sqrt{2}|x|}{\sigma})$, where σ is the standard deviation of the random variable x. The maximal amplitude of the uniform quntizer Q_0 is denoted as $x_{\max}$, which depends on the range of the input signal. The design of the quantizer Q_0 can be described in another way: firstly, the uniform quntizer for the unit standard deviation ($\sigma = 1$) is designed with the maximal amplitude $x_{\max}^{\sigma=1}$ [Jayant] and after that the denormalization is done, by dividing all thresholds and representation levels with $\triangle_0 = \frac{x_{\max}^{\sigma=1}}{x_{\max}}$, with the aim to map the range $[-x_{\max}^{\sigma=1},\ x_{\max}^{\sigma=1}]$ to the range $[-x_{\max}, x_{\max}]$. In the second step, output samples from the quantizer Q_0 (which have discrete amplitudes) are further quantized with the second quantizer Q with N levels, $N < N_0$. Input samples of the quantizer Q can take N_0 discrete values, which are equal to the output levels of the quantizer Q_0, defined with the set X. Probabilities of these discrete levels for Laplacian distribution are $P(x_i) = p(x_i)\triangle_0 = (\frac{1}{\sqrt{2}\sigma})\exp(-\frac{\sqrt{2}|x_i|}{\sigma})\ \triangle_0,\ i = 2, \dots, N_0 - 1$ and $P(x_1) = P(x_{N_0}) = (\frac{1}{2})\exp(-\frac{\sqrt{2}x_{\max}}{\sigma})$.

The aim of this paper is design of the quantizer Q. Since this quantizer is used for quantization of the samples with discrete amplitudes, its design is different from the design of the quantizers with continual amplitude samples. In this paper quantizer Q will be realized as piecewise uniform quantizer with N levels grouped in L regions. The piecewise uniform quantizer can be considered as the generalized quantizer, since for $L = N$ the nonuniform quantizer is obtained and for $L = 1$ the uniform quantizer is obtained. Each region has $M = N/L$ uniform output levels. N_i denotes the number of the input levels from the set X which belong to the ith region. Then, it holds that $N_0 = \sum_{i=1}^{L} N_i$. μ_i is the parameter of the ith region which represents the number of input levels that are mapped to one output level, in the ith region. φ_i, $i = 0, \dots, L$, denote the boundaries between regions. The design of the piecewise uniform quantizer Q will be done in the following way. Firstly, the piecewise uniform quntizer with N levels for the unit standard deviation ($\sigma = 1$) is designed. The boundaries between the regions, denoted as $\varphi_i^{\sigma=1},\ \ i = 0, \dots, L$ are obtained as $\varphi_i^{\sigma=1} = t_{iM}$, where $t_j,\ j = 0, \dots, N$ represent the thresholds of the optimal companding quntizer with N levels for $\sigma = 1$ which are given with expression (4) in the paper Peric *et al.* (2009), which was derived for the optimal compression function defined in Judell and Scharf (1986), Na (2004). In this way, the following expressions for $\varphi_i^{\sigma=1}$ are obtained:

$$\varphi_i^{\sigma=1} = \frac{3}{\sqrt{2}}\log\left(\frac{2iM + (N-2iM)\exp(-\frac{\sqrt{2}}{3}t_{\max}^{\sigma=1}(N))}{N}\right),$$
$$0 \leqslant i \leqslant L/2, \tag{1}$$
$$\varphi_i^{\sigma=1} = \frac{3}{\sqrt{2}}\log\left(\frac{N}{2N - 2iM + (2iM-N)\exp(-\frac{\sqrt{2}}{3}t_{\max}^{\sigma=1}(N))}\right),$$
$$L/2 < i \leqslant L. \tag{2}$$

$t_{\max}^{\sigma=1}(N)$ denotes the maximal amplitude of the optimal companding quantizer for the unit variance, which values can be found in Peric *et al.* (2009). The denormalization, by dividing $\varphi_i^{\sigma=1}$ with $\triangle_0$ is done. The range of the quantizer obtained in this way $[-t_{\max}(N), t_{\max}(N)]$, $t_{\max}(N) = t_{\max}^{\sigma=1}(N)/\triangle_0$, is different from the range $[-x_{\max}, x_{\max}]$, since $t_{\max}^{\sigma=1}(N) \neq x_{\max}^{\sigma=1}$. If $t_{\max}(N) < x_{\max}$, the overload distortion will exist. If $t_{\max}(N) > x_{\max}$, the range $[x_{\max}, t_{\max}(N)]$ will be unused, which leads to higher granular distortion. To avoid these negative effects, the corrections of the quantizer range will be done by dividing the boundaries between the regions with $t_{\max}/x_{\max}$. In this way, the boundaries between the regions φ_i are obtained. The range of the quantizer is $[-x_{\max}, x_{\max}]$.

$d_i,\ i = 1, \ldots, L$ denotes the width of the ith region of the fixed piecewise uniform quantizer Q and it can be calculated as

$$d_i = \frac{\varphi_i - \varphi_{i-1}}{M}. \tag{3}$$

The output levels of the quantizer Q, denoted with $y_{ij},\ i = 1, \ldots, L;\ j = 1, \ldots, M$ (i represents the region where the output level belongs and j represents the number of the output level within that region) can be calculated as

$$y_{ij} = t_{i-1} + \left(\frac{2j-1}{2}\right)d_i. \tag{4}$$

During the quantization process, an irreversible error is made, which is defined with distortion D. Since input samples have discrete amplitudes which are limited with $x_{\max}$, the fixed quantizer Q has only the granular distortion D_g, which is defined as:

$$D = D_g = \sum_{i=1}^{L}\sum_{j=1}^{M}\sum_{k=1}^{\mu_i}(x_{ijk} - y_{ij})^2 P(x_{ijk}), \tag{5}$$

where $x_{ijk} \in X$ is one of μ_i input levels which are mapped to the output level y_{ij}.

The quality of the quantized signal is usually defined with the signal-to-quantization noise ratio *SQNR*, which is defined in the following way:

$$\mathit{SQNR}\ [\mathrm{dB}] = 10\log_{10}\left(\frac{\sigma^2}{D_g}\right), \tag{6}$$

where σ^2 is a variance of the input signal with discrete amplitudes.

Table 1

SQNR values of the piecewise uniform quantizer for the discrete input samples for different values of N and L

	SQNR [dB]		
	$N = 16$, $t_{\max}^{\sigma=1}(N) = 6.01$	$N = 32$, $t_{\max}^{\sigma=1}(N) = 7.41$	$N = 64$, $t_{\max}^{\sigma=1}(N) = 8.85$
$L = 2$	10.90	16.427	23.071
$L = 4$	14.962	20.90	26.098
$L = 8$	16.652	22.826	27.828
$L = 16$	17.417	23.454	29.069
$SQNR_{\mathrm{opt}}$ [dB]	18.080	23.840	29.727

In Table 1, values of *SQNR* for piecewise uniform quantizer are given, for different values of N and L, for the input signal with Laplacian distribution with unit variance ($\sigma^2 = 1$). Input samples have discrete amplitudes which have been already quantized with the uniform quantizer with $N_0 = 512$ levels and $x_{\max}^{\sigma=1} = 7.9$. In the last row of Table 1, values of *SQNR* for the optimal nonuniform quantizer for samples with continual amplitudes is given, denoted with $SQNR_{\mathrm{opt}}$. These values are taken from the paper Peric *et al.* (2009). Based on the results shown in Table 1, we can conclude that if the number of regions L ($L \leqslant N$) increases, performances of piecewise uniform quantizer for the discrete input became more closer to the performances of the optimal nonuniform quantizer for the continual input.

3. Design of the Adaptive Piecewise Uniform Quantizer for Discrete Input Samples

In this section, the forward adaptation of the piecewise uniform quantizer for discrete input samples will be done. The adaptation is done on the frame-by-frame (or, block-by-block for the image compression) basis. The standard deviation σ of the frame is found. Quantized standard deviation of the frame $\widehat{\sigma}$ is transmitted to the receiver. The maximal amplitude of the adaptive quantizer before denormalization, denoted as $t_{\max}^{\mathrm{adapt}}$, is defined as:

$$t_{\max}^{\mathrm{adapt}} = k\widehat{\sigma}, \tag{7}$$

where k is the loading factor of the adaptive quantizer, whose optimal value is found numerically, as it will be described in the Section 5.2. The boundaries between the regions $\varphi_i^{\mathrm{adapt}}$ of the piecewise uniform adaptive quantizer is defined with the following expressions:

$$\varphi_i^{\mathrm{adapt}} = \frac{3\widehat{\sigma}}{\sqrt{2}\,\triangle_0}\log\left(\frac{2iM + (N - 2iM)\exp\left(-\frac{\sqrt{2}}{3}\frac{t_{\max}^{\mathrm{adapt}}}{\widehat{\sigma}}\right)}{N}\right),$$
$$0 \leqslant i \leqslant L/2, \tag{8}$$

$$\varphi_i^{\text{adapt}} = \frac{3\widehat{\sigma}}{\sqrt{2}\,\triangle_0}\log\left(\frac{N}{2N-2iM+(2iM-N)\exp(-\frac{\sqrt{2}}{3}\frac{t_{\max}^{\text{adapt}}}{\widehat{\sigma}})}\right),$$
$$L/2 < i \leqslant L. \tag{9}$$

Division with $\triangle_0$ in the previous expressions is done due to denormalization. The maximal amplitude of the adaptive piecewise uniform quantizer is $x_{\max}^{\text{adapt}} = t_{\max}^{\text{adapt}}/\triangle_0$ (after denormalization). $d_i^{\text{adapt}},\ i = 1,\dots,L$, which denotes the width of the ith region of the piecewise uniform adaptive quantizer, is defined as:

$$d_i^{\text{adapt}} = \frac{\varphi_i^{\text{adapt}} - \varphi_{i-1}^{\text{adapt}}}{M}. \tag{10}$$

The output levels of this quantizer, denoted as $y_{ij}^{\text{adapt}},\ i = 1,\dots,L;\ j = 1,\dots,M$, are defined as:

$$y_{ij}^{\text{adapt}} = \varphi_{i-1}^{\text{adapt}} + \left(\frac{2j-1}{2}\right)d_i^{\text{adapt}}. \tag{11}$$

Using adaptation, we adjust the amplitude range of the quantizer to the variance of the input signal. In Fig. 1, the range $I = (-x_{\max}, x_{\max})$ of the uniform quantizer Q_0, which precedes to the adaptive piecewise uniform quantizer, is shown firstly. Recall that input samples of the adaptive quantizer are bounded in amplitude with $x_{\max}$. Also, in Fig. 1, the range $I^{\text{adapt}} = (-x_{\max}^{\text{adapt}}, x_{\max}^{\text{adapt}})$ of the adaptive piecewise uniform quantizer is shown for two cases: when $x_{\max}^{\text{adapt}} < x_{\max}$ (i.e., I^{adapt} becomes narrower than I) and when $x_{\max}^{\text{adapt}} > x_{\max}$ (i.e., I^{adapt} becomes wider than I). The case when $x_{\max}^{\text{adapt}} > x_{\max}$ will be deeply analyzed. In this case, since the input samples are amplitudely bounded by $x_{\max}$, the representation levels in the range $(-x_{\max}^{\text{adapt}}, -x_{\max}) \cup (x_{\max}, x_{\max}^{\text{adapt}})$ is unused. But, as it was said earlier, input samples are bounded in amplitude with $x_{\max}$. Therefore, not all N representation levels are used, but only N_1 (N_1 is some number smaller than N). Because of that, higher distortion (i.e., lower *SQNR*) is obtained, compared to the

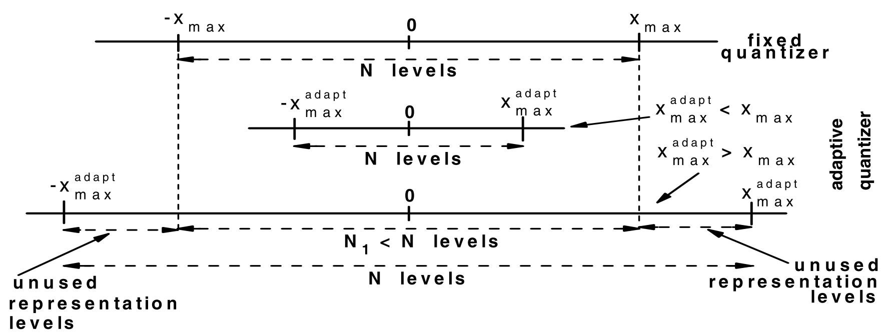

Fig. 1. The range of the fixed and the forward adaptive quantizers.

fixed quantizer (without adaptation). We can conclude that if adaptation is done when $x_{\max}^{\text{adapt}} > x_{\max}$, worse performances are obtained. Therefore, adaptation should not be done in this case. We can introduce the following rule: adaptation should be done only when $x_{\max}^{\text{adapt}} < x_{\max}$. If this condition is not fulfilled, the fixed quantizer should be used. This is the main difference between adaptive quantizers for continual and for discrete input samples, because for continual input samples adaptation gives good results in both cases: $x_{\max}^{\text{adapt}} < x_{\max}$ and $x_{\max}^{\text{adapt}} > x_{\max}$.

For the adaptive quantizer, input levels can take values from the range $(-x_{\max}, x_{\max})$ and the output values can take values from the range $(-x_{\max}^{\text{adapt}}, x_{\max}^{\text{adapt}})$, $x_{\max}^{\text{adapt}} < x_{\max}$. The input levels from the range $(x_{\max}^{\text{adapt}}, x_{\max})$, whose number is denoted as μ_{L+1}, are mapped into the last output level y_{LM}, which leads to the existence of the overload distortion D_{ov}. Therefore, for the adaptive quantizer, we have the granular distortion given with the expression (5) and the overload distortion given with the following expression:

$$D_{\text{ov}} = 2\sum_{k=1}^{\mu_{L+1}} (x_{\text{LMk}} - y_{\text{LM}})^2 P(x_{\text{LMk}}). \tag{12}$$

The total distortion is equal to the sum of these two distortions, i.e.,:

$$D = D_g + D_{\text{ov}}. \tag{13}$$

4. Application of the Previous Models on the Grayscale Images

In this section, a weighting function for *PSQNR* calculation is introduced and an algorithm for image processing is presented.

4.1. *Weighting Function*

As a measure of the quality of the piecewise uniform quantizer *Q* (fixed or adaptive) we will use the peak signal-to-quantization noise ratio for the block B of the original image pixels, defined as

$$\textit{PSQNR}\ [\text{dB}] = 10\log_{10}\left(\frac{x_{\max}^2}{D}\right) = 10\log_{10}\left(\frac{x_{\max}^2}{D_d}\right) \ . \tag{14}$$

Since the distortion D_d depends on σ_d, it follows that *PSQNR* [dB] also depend on σ_d which can take values from 1 to $x_{\max}$ ($x_{\max} = 255$). Actually, σ_d also can take value 0, but this case will not be considered since if $\sigma_d = 0$ then block B_d does not contain any information (in this case x_{av} carries all information about block B). σ_d can be written in the logarithmic domain as $\sigma_d\ [\text{dB}] = \widehat{\sigma}_d\ [\text{dB}] = 20\log_{10}\frac{\sigma_d}{\sigma_0}$, where σ_0 is some referent standard deviation. Without losing a generality, we can take that $\sigma_0 = 255$. Then σ_d [dB] can take values from $20\log_{10}\frac{1}{255} = -48.13$ dB to $20\log_{10}\frac{255}{255} = 0$ dB. As a measure of the quality of the reconstructed block of the image, we will use the average *PSQNR* in the range $[-48.13\ \text{dB},\ 0\ \text{dB}]$ of σ_d [dB]. The averaging will be done in two ways:

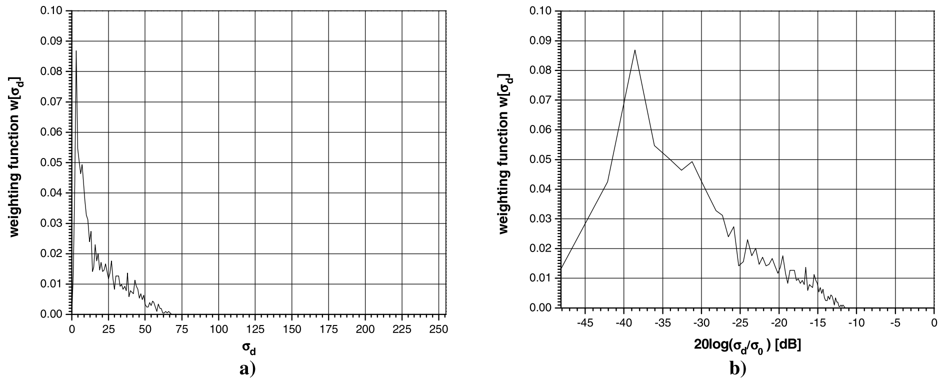

Fig. 2. Weighting function: (a) in the linear domain and (b) in the logarithmic domain.

1. All values of σ_d [dB] in that range have the same weight. In that way we obtained *PSQNR*$_{\mathrm{av}}$.

$$\mathit{PSQNR}_{\mathrm{av}} = \frac{1}{255} \sum_{\sigma_{d_i}=1}^{255} PSQNR(\widehat{\sigma}_{d_i}) \text{ [dB]}. \tag{15}$$

2. Since all of the input samples do not occur with the same probabilities we introduce weighting function for σ_d [dB]. This weighting function is calculated based on the three standard test images (Lena, Boat and Airplane) and it is shown in Fig. 2a in the linear domain and in Fig. 2b in the logarithmic domain. In this way *PSQNR*$_{\mathrm{wav}}$ is obtained.

$$\mathit{PSQNR}_{\mathrm{wav}} = \sum_{\sigma_{d_i}=1}^{255} w\big(\widehat{\sigma}_{d_i}\big) PSQNR\big(\widehat{\sigma}_{d_i}\big) \text{ [dB]}. \tag{16}$$

4.2. *An Algorithm for the Image Processing*

In this section, we will explain the application of the previously described quantizers (fixed and adaptive) on grayscale images. Pixels of the image can take integer values from 0 to $x_{\max}$. We consider the case whit $x_{\max} = 255$ (each pixel is represented with 8 bits). The algorithm for the image processing starts with dividing the image into blocks, whose size is $m \times m$ (we will use $m = 4$). Each block is processed separately. Now, an algorithm will be described, where the fixed piecewise uniform quantizer is used for the processing of one arbitrary block, which is denoted with B and whose pixels are denoted with $x_{i,j},\ i = 1, \ldots, m;\ j = 1, \ldots, m$.

1. The average value x_{av} of the block B is found, this value is quantized and obtained quantized average value $\hat{x}_{\mathrm{av}}$ is transmitted to the receiver.

2. The *difference block* B_d, which size is $m \times m$ is formed. Elements of that block, denoted with $d_{i,j},\ i = 1, \ldots, m;\ j = 1, \ldots, m$ are obtained as the difference of the pixel values of the block B and the quantized mean value $\hat{x}_{\mathrm{av}}$, i.e.,

$$d_{i,j} = x_{i,j} - \hat{x}_{\mathrm{av}}. \tag{17}$$

   Elements $d_{i,j}$ of the block B_d can take integer values from the interval $[-x_{\max}, x_{\max}]$. $d_{i,j}$ are integer because they are obtained as a difference between two integer values. Elements of the block B_d have the Laplacian distribution (Jayant, 1984).
3. Since the elements of the block B_d are integers (i.e., they have discrete amplitudes), fixed quantizer designed for discrete input should be used. Quantized elements of the block B_d, denoted with $\hat{d}_{i,j}$ are binary coded with R bits ($\log_2 N$) and transmitted to the receiver. The distortion which is made by quantization of the elements $d_{i,j}$ can be expressed as:

$$D_d = \frac{1}{m * m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(d_{i,j} - \hat{d}_{i,j}\right)^2. \tag{18}$$

4. In the receiver, after the reception of $\hat{x}_{\mathrm{av}}$ and $\hat{d}_{i,j}$, the reconstruction of the pixels of the original image is done as

$$\hat{x}_{i,j} = \hat{d}_{i,j} + \hat{x}_{\mathrm{av}}. \tag{19}$$

   The reconstructed block of the image, denoted as $\widehat{\mathrm{B}}$ consists of the reconstructed pixels $\hat{x}_{i,j}$. The difference between the original block B and the reconstructed block $\widehat{\mathrm{B}}$ is measured by distortion $D = \frac{1}{m*m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(x_{i,j} - \hat{x}_{i,j}\right)^2$. Based on (14), (15) and (16), it follows that

$$\begin{aligned} D &= \frac{1}{m * m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(x_{i,j} - \hat{x}_{i,j}\right)^2 \\ &= \frac{1}{m * m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(d_{i,j} - \hat{d}_{i,j}\right)^2 = D_d. \end{aligned} \tag{20}$$

For the adaptive quantizer, the previous algorithm with the following changes is used. The Step 2 is extended with the Step 2.1, where the standard deviation σ_d of the block B_d is calculated as $\sigma_d = \left[\sqrt{\frac{1}{m*m} \sum_{i=1}^{m} \sum_{j=1}^{m} \left(x_{i,j} - \hat{x}_{\mathrm{av}}\right)^2}\right]$ and transmited to the receiver. $[x]$ denotes the nearest integer for x. After that, in the Step 3, the adaptive piecewise uniform quantizer, designed for σ_d, is used instead of the fixed quantizer.

In Table 2, lower (x_i^{low}) and upper (x_i^{up}) boundaries of intervals of the fixed and the adaptive quantizers used in the Step 3 for quantization of the block B_d are shown, for $N = 16$, $L = 16$ and for different values of σ_d. Due to the symmetry, only the positive

Table 2

Boundaries of the intervals (thresholds) for the fixed and the adaptive piecewise uniform quantizer (designed for discrete input samples), applied on the image compression, for $N = 16$, $L = 16$ and for different values of σ_d

	Fixed quantizer			Adaptive quantizer								
	$\sigma_d = 32$			$\sigma_d = 2$			$\sigma_d = 32$			$\sigma_d = 100$		
i	x_i^{low}	x_i^{up}	d_i	x_i^{low}	x_i^{up}	d_i	x_i^{low}	x_i^{up}	d_i	x_i^{low}	x_i^{up}	d_i
1	0	16	16	0	0	0	0	0	0	0	19	19
2	17	32	15	1	1	0	1	1	0	20	42	22
3	33	48	15	2	2	0	2	2	0	43	67	24
4	49	60	11	3	3	0	3	3	0	68	94	26
5	61	108	47	4	6	2	4	35	31	95	126	31
6	109	156	47	7	9	2	36	67	31	127	163	36
7	157	204	47	10	12	2	68	99	31	164	207	43
8	205	255	50	13	15	2	100	126	26	208	255	47

Table 3

Boundaries of the intervals (thresholds) for the fixed and the adaptive piecewise uniform quantizer (designed for continual input samples), applied on the image compression, for $N = 16$, $L = 16$ and for different values of σ_d

	Fixed quantizer			Adaptive quantizer								
	$\sigma_d = 32$			$\sigma_d = 2$			$\sigma_d = 32$			$\sigma_d = 100$		
i	x_i^{low}	x_i^{up}	d_i	x_i^{low}	x_i^{up}	d_i	x_i^{low}	x_i^{up}	d_i	x_i^{low}	x_i^{up}	d_i
1	0	9	9	0	1	1	0	8	8	0	27	27
2	10	18	8	2	2	0	9	19	10	28	58	30
3	19	30	11	3	4	1	20	32	12	59	94	35
4	31	43	12	5	6	1	33	46	13	95	138	43
5	44	61	17	7	8	1	47	64	17	139	192	53
6	62	84	22	9	10	1	65	88	23	193	265	72
7	85	119	34	11	13	2	89	124	35	266	374	108
8	120	194	74	14	19	5	125	199	74	375	608	233

parts of the quantizers are shown. Also, the widths of the intervals $d_i = x_i^{\text{up}} - x_i^{\text{low}}$ are given. Since the pixels of the images are integers, boundaries of the intervals are also integers. For some intervals, lower and upper boundaries are the same. These intervals have only one representation level, which is equal to the boundaries and the widths of these intervals d_i are equal to 0. This case is specific for the discrete source; it does not exist for the continual source.

In Table 3, decision thresholds x_i^{low} and x_i^{up} and widths of the intervals d_i are given, for fixed and adaptive piecewise uniform quantizers designed for the continual input samples. Due to the symmetry, only the positive parts of the quantizers are shown. Now, we will consider the case when σ_d =100. From Table 3, we can see that the range of the

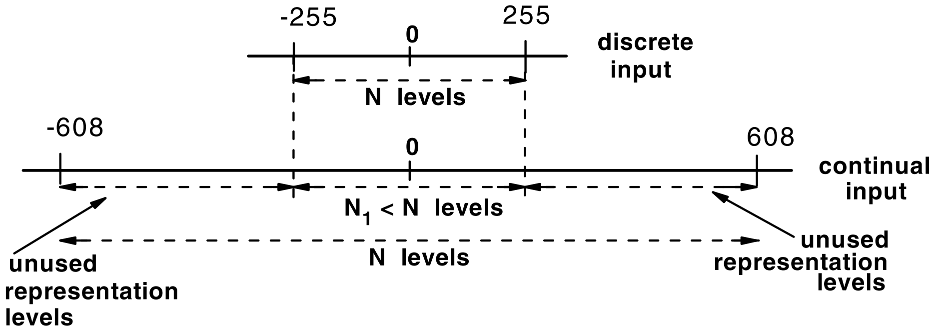

Fig. 3. The range of the forward adaptive piecewise uniform quantizers designed for continual and for discrete input for $\sigma_d = 100$.

adaptive quantizer designed for continual input is $[-608, 608]$, which is much wider than the range of the possible input values of discrete signal $[-255, 255]$. It means that the levels in the range $[-608, -255] \cup [255, 608]$ are unused, i.e., the number of levels N_1 in the range $[-255, 255]$ which are effectively used is smaller than total number of levels N. This is shown in Fig. 3. In general, the same phenomenon occurs for large values of σ_d. On the other side, the range of the adaptive quantizer for discrete input is limited with the maximal input value (in our case $[-255, 255]$) and therefore all N representation levels are effectively used. Due to unused levels, adaptive quantizer designed for continual input samples has smaller *PSQNR* compared to the adaptive quantizer designed for discrete input samples, which will be experimentally proved in the next section.

5. Numerical and Experimental Results

In this section numerical and experimental results are presented for fixed and adaptive piecewise uniform quantizer for discrete and continual input samples.

5.1. *Numerical Results for the Fixed Piecewise Uniform Quantizer*

In Fig. 4, theoretically obtained (using the expression (5) for the distortion) dependences of *PSQNR* on σ_d [dB] are shown for $N = 16$ and $N = 32$, for the fixed piecewise uniform quantizer for discrete input samples. Two average values of *PSQNR* ($PSQNR_{\mathrm{av}}$ and $PSQNR_{\mathrm{wav}}$) are calculated in the way described in the Section 4.1 and these values are given in Fig. 4. We can see that these values are quite different.

In Table 4, experimental results, obtained by applying the fixed piecewise uniform quantizer for discrete input samples for image compression (in Step 3 of the algorithm in Section 4.1) of three previously mentioned standard test images, for different values of N and L are shown. Values in Table 4 are averaged values for these three images. We can see that experimentally obtained values of *PSQNR* (from Table 4) are matched very well to theoretically obtained values of $PSQNR_{\mathrm{wav}}$ (in Fig. 4), while they are quite different

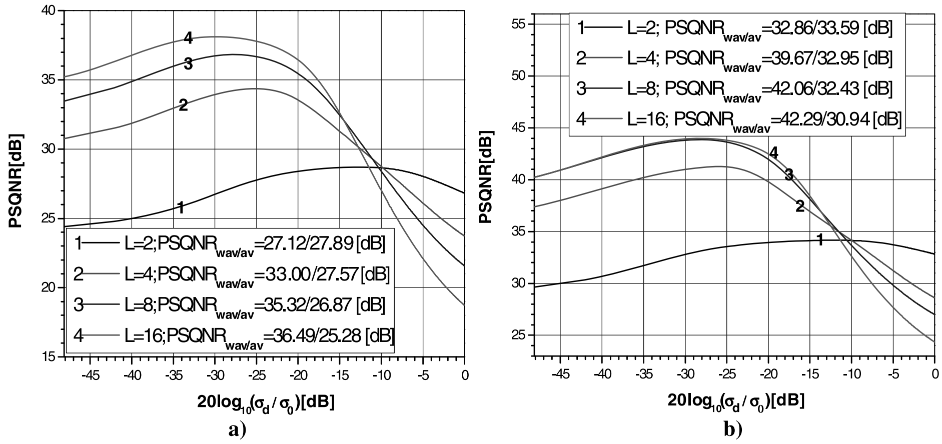

Fig. 4. *PSQNR* of fixed piecewise uniform quantizer for discrete input for: (a) $N = 16$ and (b) $N = 32$.

Table 4

Experimental results for the fixed piecewise uniform quantizer designed for discrete input samples

	$N = 16$			$N = 32$			
	$L = 2$	$L = 4$	$L = 8$	$L = 2$	$L = 4$	$L = 8$	$L = 16$
PSQNR [dB]	25.358	33.037	35.817	33.272	40.058	42.380	42.640

from values of *PSQNR*$_{\rm av}$ (in Fig. 4). It proves that the theory obtained in Section 2 is correct. Also, it justifies the introduction of the weighting function for σ_d and shows that *PSQNR*$_{\rm wav}$ should be used instead of *PSQNR*$_{\rm av}$.

Experimental results for the fixed piecewise uniform quantizer designed for continual input samples are similar with the results in Table 4, since the value of σ_d which was used ($\sigma_d = 32$) is adjusted to the test images.

5.2. *Numerical Results for the Adaptive Piecewise Uniform Quantizer*

In this section, numerical results for the adaptive piecewise uniform quantizer for discrete input samples are given. In Fig. 5, theoretically obtained (using the expressions (5), (12) and (13) for the distortion) dependences of *PSQNR* on σ_d [dB] are shown for $N = 16$ and $N = 32$. *PSQNR*$_{\rm wav}$ is also shown in this figure. *PSQNR*$_{\rm av}$ is not considered, due to the conclusion in the previous section. The optimal values of the parameter k, which are numerically found to maximize *PSQNR*$_{\rm wav}$, are also given in Fig. 5.

From Fig. 5 we can see an interesting effect: *PSQNR* for adaptive quantizers increases for small σ_d [dB]. Now, we will explain this effect considering one example with parameters $N_0 = 256$, $N = 32$ and $L = 8$. For the fixed quantizer, we choose 32 output levels from the set of 256 input levels, i.e., matching between input and output levels is 12.5%. For the adaptive quantizer, if σ_d is small, $x_{\max}^{\rm adapt}$ also will be small. We consider the case:

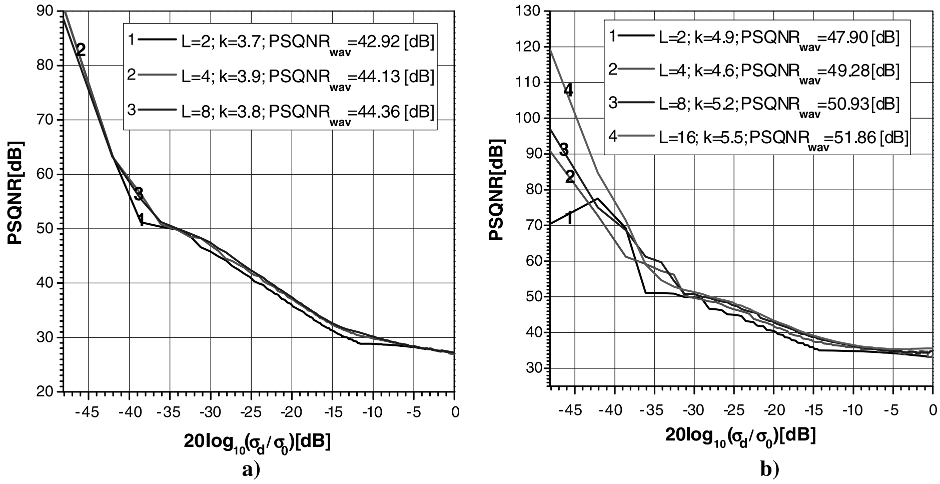

Fig. 5. *PSQNR* of adaptive piecewise uniform quantizer for discrete input for: (a) $N = 16$ and (b) $N = 32$.

Table 5

Experimental results for the adaptive piecewise uniform quantizer designed for discrete and continual input samples

	$N = 16$			$N = 32$			
	$L = 2$	$L = 4$	$L = 8$	$L = 2$	$L = 4$	$L = 8$	$L = 16$
PSQNR [dB]	45.029	45.483	45.785	49.998	50.221	50.310	51.125
PSQNR$_{\rm cont}$ [dB]	41.929	42.883	44.345	46.548	47.085	48.685	49.665

$x_{\max}^{\rm adapt} = \frac{1}{4}x_{\max}$. Therefore, inside the range of the adaptive quantizer $(-x_{\max}^{\rm adapt}, x_{\max}^{\rm adapt})$ there are $N_0/4 = 64$ input levels and among them we choose $N = 32$ output levels. So, for the adaptive quantizer, matching between input and output levels is 50%. If some input level is equal to some output level, distortion for that input level is zero. Therefore, if percentage of matching between input and output levels increases then distortion decreases and then *PSQNR* increases. It is clear from the above example that matching percentage increases when σ_d decreases.

In Table 5, experimental results for the adaptive piecewise uniform quantizer for discrete input samples are given, for different values of N and L, obtained by applying this quantizer in Step 3 of the algorithm in Section 4.2. Values in Table 5 are averaged values for standard test images (Lena, Street and Boat). We can see that experimentally obtained values of *PSQNR* in Table 5 are matched very well to theoretically obtained values of *PSQNR*$_{\rm wav}$ in Fig. 5. It proves the theory from Section 3.

PSQNR values in Table 5 are very high, which means that high quality reconstructed images are obtained, i.e., near lossless compression is achieved using this adaptive quantizer.

In the aim of comparison, the average values of *PSQNR* (denoted as $PSQNR_{\mathrm{cont}}$) obtained by application of the adaptive piecewise uniform quantizer designed for continual input samples of three above mentioned standard test images are also given in Table 5. We can see that the adaptive quantizer designed for discrete input samples gives higher *PSQNR* for 1.46 to 3.45 dB compared the adaptive quantizer designed for continual input samples, because of the reason explained at the end of Section 4.2.

6. Conclusion

Design of fixed and adaptive piecewise uniform quantizers for discrete input samples with Laplacian distribution was considered in this paper. These quantizers have a great practical importance since they are used for compression of already quantized signals. Piecewise uniform quantizer was analyzed since it could be considered as generalized quantizer. It was shown that the design of quantizers for discrete input samples is different from the design of quantizers for continual input samples. Since discrete input samples are limited in amplitude, the overload distortion does not exist. Adaptation of these quantizers should be done only under one condition: the maximal amplitude of the adaptive quantizer should be smaller than the maximal amplitude of the fixed quantizer. Also, for the adaptive quantizer, *PSQNR* increases for small input variances. An algorithm for compression of grayscale images was presented, applying these quantizers for discrete input samples. A weighting function for the standard deviation of the difference of the pixel value and the quantized mean value of the block which pixels belong was introduced. Average *PSQNR* was calculated in two ways: by classic averaging (without using weighting function) and by weighted averaging (using weighting function). Experiments were done, applying these fixed and adaptive quantizers for compression of three test grayscale images (Lena, Street and Boat). It was shown that experimentally obtained *PSQNR* was matched very well with theoretically obtained weighted *PSQNR*. In this way, developed theory was proved. Also, it was shown that the average *PSQNR* obtained by using the weighting function should be used instead of the average *PSQNR* obtained without using the weighting function. We have compared adaptive piecewise uniform quantizer designed for discrete input samples with fixed piecewise uniform quantizer designed for discrete input samples, in order to show that higher *PSQNR* of 9 to 20 dB is obtained. Also it is shown that the adaptive piecewise uniform quantizer designed for discrete input samples gives higher *PSQNR* for 1.46 to 3.45 dB compared the adaptive piecewise uniform quantizer designed for continual input samples. Capitalizing on this, the conclusion arises, that the discrete model is more appropriate for image quantization than continual model. With the adaptive piecewise uniform quantizer, the near lossless compression was achieved.

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Kodavimo algoritmas pustoniams vaizdams, grindžiamas atkarpomis tiesiniu, pastoviuoju dydžių keitikliu

Milan S. SAVIĆ, Zoran H. PERIĆ, Milan R. DINČIĆ

Šiame straipsnyje atkarpomis tiesinis, pastovusis dydžių keitiklis yra sudarytas ir išanalizuotas Laplaso šaltinio įėjimo atskaitoms su diskrečiosiomis amplitudėmis. Gauta jo tolimesnė adaptacija. Tokio tipo keitikliai yra labai dažnai taikomi praktikoje jau kvantuotų signalų glaudinimui bei kodavimui. Parodyta, kad šių keitiklių sudarymas bei jų adaptacija diskrečioms įėjimo atskaitoms skiriasi nuo keitiklių sudarymo bei jų adaptacijos tolydinėms įėjimo atskaitom. Pateikta svorinė PSQNR (didžiausios signalo ir kvantavimo triukšmo reikšmių santykis) svorinė funkcija, kuri gauta taikant standartinio testinio vaizdo nuokrypio tikimybinį tankį. Atlikti eksperimentai, šiais keitikliais glaudinant pustonius vaizdus. Eksperimentų rezultatai patvirtina teorinius rezultatus. Adaptyvusis atkarpomis tiesinis, pastovusis keitiklis, sukurtas diskrečioms įėjimo atskaitoms, duoda nuo 9 iki 20 dB aukštesnį PSQNR lyginant su fiksuotuoju atkarpomis tiesiniu keitikliu, sudarytų diskrečiosioms įėjimo atskaitoms. Taip pat yra parodyta, kad adaptyvusis atkarpomis tiesinis, pastovusis keitiklis, sudarytas diskrečioms įėjimo atskaitoms duoda nuo 1.46 iki 3.45 aukštesnį PSQNR, nei adaptyvusis atkarpomis tiesinis keitiklis, sudarytas tolydinėms įėjimo atskaitoms. Šis faktas įrodo, kad diskretusis modelis esti labiau tinkamas vaizdo kvantavimui nei tolydinis modelis.

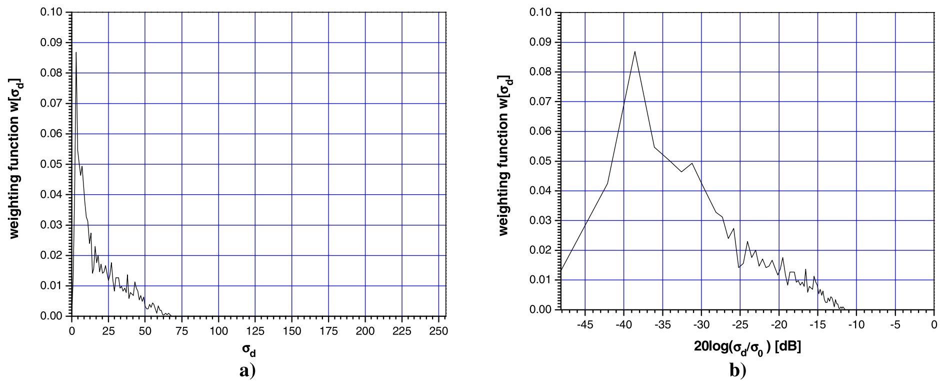


Fig. 2. Weighting function: (a) in the linear domain and (b) in the logarithmic domain.

1. All values of σ_d [dB] in that range have the same weight. In that way we obtained $PSQNR_{\mathrm{av}}$.

$$PSQNR_{\mathrm{av}} = \frac{1}{255}\sum_{\sigma_{d_i}=1}^{255} PSQNR(\widehat{\sigma}_{d_i})\ [\mathrm{dB}]. \tag{15}$$

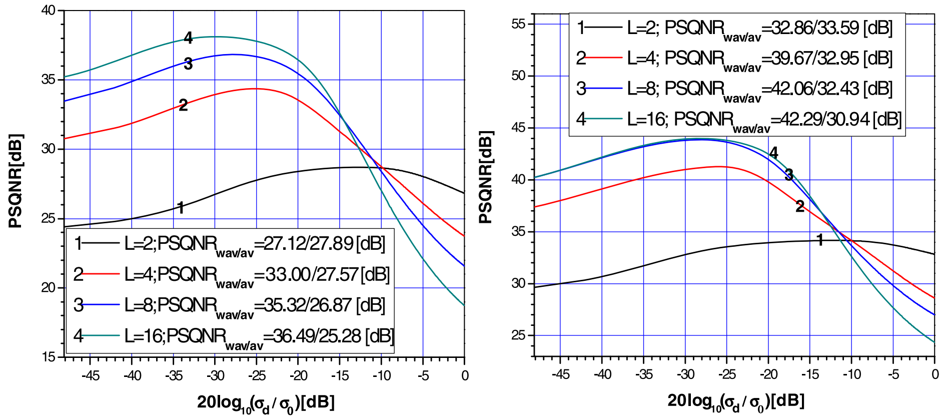
2. Since all of the input samples do not occur with the same probabilities we introduce weighting function for σ_d [dB]. This weighting function is calculated based on the three standard test images (Lena, Boat and Airplane) and it is shown in Fig. 2a in the linear domain and in Fig. 2b in the logarithmic domain. In this way $PSQNR_{\mathrm{wav}}$ is obtained.

$$PSQNR_{\mathrm{wav}} = \sum_{\sigma_{d_i}=1}^{255} w\big(\widehat{\sigma}_{d_i}\big) PSQNR\big(\widehat{\sigma}_{d_i}\big)\ [\mathrm{dB}]. \tag{16}$$

4.2. *An Algorithm for the Image Processing*

In this section, we will explain the application of the previously described quantizers (fixed and adaptive) on grayscale images. Pixels of the image can take integer values from 0 to $x_{\max}$. We consider the case whit $x_{\max} = 255$ (each pixel is represented with 8 bits). The algorithm for the image processing starts with dividing the image into blocks, whose size is $m \times m$ (we will use $m = 4$). Each block is processed separately. Now, an algorithm will be described, where the fixed piecewise uniform quantizer is used for the processing of one arbitrary block, which is denoted with B and whose pixels are denoted with $x_{i,j},\ i = 1, \ldots, m;\ j = 1, \ldots, m$.

1. The average value x_{av} of the block B is found, this value is quantized and obtained quantized average value $\hat{x}_{\mathrm{av}}$ is transmitted to the receiver.



a) b)

Fig. 4. *PSQNR* of fixed piecewise uniform quantizer for discrete input for: (a) $N = 16$ and (b) $N = 32$.

Table 4

Experimental results for the fixed piecewise uniform quantizer designed for discrete input samples

	$N = 16$			$N = 32$			
	$L = 2$	$L = 4$	$L = 8$	$L = 2$	$L = 4$	$L = 8$	$L = 16$
PSQNR [dB]	25.358	33.037	35.817	33.272	40.058	42.380	42.640

from values of *PSQNR*$_{\mathrm{av}}$ (in Fig. 4). It proves that the theory obtained in Section 2 is correct. Also, it justifies the introduction of the weighting function for σ_d and shows that *PSQNR*$_{\mathrm{wav}}$ should be used instead of *PSQNR*$_{\mathrm{av}}$.

Experimental results for the fixed piecewise uniform quantizer designed for continual input samples are similar with the results in Table 4, since the value of σ_d which was used ($\sigma_d = 32$) is adjusted to the test images.

5.2. *Numerical Results for the Adaptive Piecewise Uniform Quantizer*

In this section, numerical results for the adaptive piecewise uniform quantizer for discrete input samples are given. In Fig. 5, theoretically obtained (using the expressions (5), (12) and (13) for the distortion) dependences of *PSQNR* on σ_d [dB] are shown for $N = 16$ and $N = 32$. *PSQNR*$_{\mathrm{wav}}$ is also shown in this figure. *PSQNR*$_{\mathrm{av}}$ is not considered, due to the conclusion in the previous section. The optimal values of the parameter k, which are numerically found to maximize *PSQNR*$_{\mathrm{wav}}$, are also given in Fig. 5.

From Fig. 5 we can see an interesting effect: *PSQNR* for adaptive quantizers increases for small σ_d [dB]. Now, we will explain this effect considering one example with parameters $N_0 = 256$, $N = 32$ and $L = 8$. For the fixed quantizer, we choose 32 output levels from the set of 256 input levels, i.e., matching between input and output levels is 12.5%. For the adaptive quantizer, if σ_d is small, $x_{\max}^{\mathrm{adapt}}$ also will be small. We consider the case:

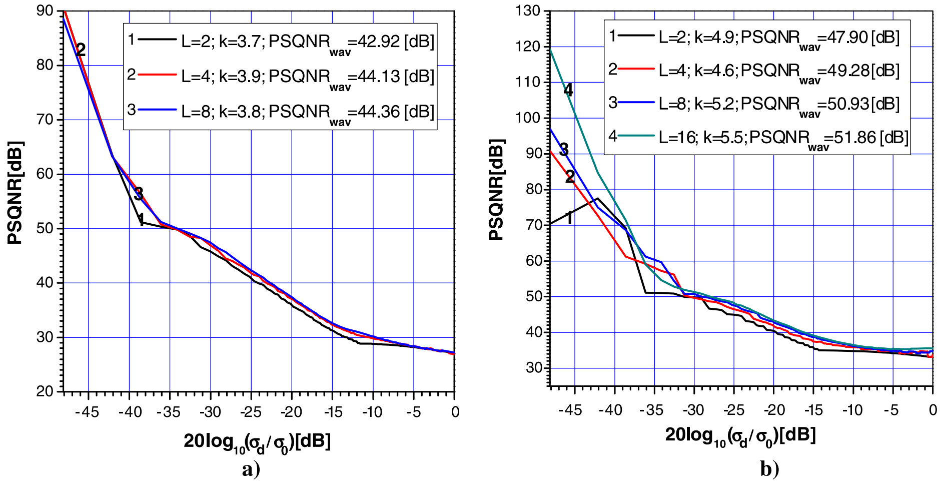


Fig. 5. *PSQNR* of adaptive piecewise uniform quantizer for discrete input for: (a) $N = 16$ and (b) $N = 32$.

Table 5

Experimental results for the adaptive piecewise uniform quantizer designed for discrete and continual input samples

	$N = 16$			$N = 32$			
	$L = 2$	$L = 4$	$L = 8$	$L = 2$	$L = 4$	$L = 8$	$L = 16$
PSQNR [dB]	45.029	45.483	45.785	49.998	50.221	50.310	51.125
PSQNR$_{\mathrm{cont}}$ [dB]	41.929	42.883	44.345	46.548	47.085	48.685	49.665

$x_{\max}^{\mathrm{adapt}} = \frac{1}{4}x_{\max}$. Therefore, inside the range of the adaptive quantizer $(-x_{\max}^{\mathrm{adapt}}, x_{\max}^{\mathrm{adapt}})$ there are $N_0/4 = 64$ input levels and among them we choose $N = 32$ output levels. So, for the adaptive quantizer, matching between input and output levels is 50%. If some input level is equal to some output level, distortion for that input level is zero. Therefore, if percentage of matching between input and output levels increases then distortion decreases and then *PSQNR* increases. It is clear from the above example that matching percentage increases when σ_d decreases.

In Table 5, experimental results for the adaptive piecewise uniform quantizer for discrete input samples are given, for different values of N and L, obtained by applying this quantizer in Step 3 of the algorithm in Section 4.2. Values in Table 5 are averaged values for standard test images (Lena, Street and Boat). We can see that experimentally obtained values of *PSQNR* in Table 5 are matched very well to theoretically obtained values of *PSQNR*$_{\mathrm{wav}}$ in Fig. 5. It proves the theory from Section 3.